P H Y S I C A L   R E V I E W   L E T T E R S

Operational Multipartite Entanglement Measures

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We introduce two operational entanglement measures that are applicable for arbitrary multipartite (pure or mixed) states. One of them characterizes the potentiality of a state to generate other states via local operations assisted by classical communication and the other characterizes the simplicity of generating the state at hand. We show how these measures can be generalized to two classes of entanglement measures. Moreover, we compute the new measures for pure few-partite systems and use them to characterize the entanglement contained in a three-qubit state. We identify the Greenberger-Horne-Zeilinger and W state as the most powerful pure three-qubit states regarding state manipulation.

Entanglement is of paramount importance in many fields of science. Because of its existence, applications such as teleportation, quantum computation, quantum simulation, and quantum error correction, to name a few, are feasible [1]. Moreover, the application of entanglement theory in other fields of science, most prominently condensed matter physics, has opened new routes towards the understanding of quantum many-body systems [2]. Because of its importance, an enormous effort has been made to qualify and quantify multipartite entanglement. Different entanglement classes have been identified and several entanglement measures have been introduced [3]. Some of them originated from analyzing the potentiality of a state for a particular realization of an application, such as the localizable entanglement [4], others arose from the generalization of classical correlation measures, such as the generalization of the squashed entanglement [3,5].

Despite these results, we are still far from completely understanding multipartite entanglement. The lack of knowledge stems, on the one hand, from the fact that the number of nonlocal parameters scales exponentially with the number of subsystems and, on the other hand, from the fact that the operations that are central in the investigation of entanglement, the local operations assisted by classical communication (LOCC), are notoriously difficult to analyze in general [6]. The importance of LOCC in this context is due to the fact that LOCC corresponds to those operations which can be implemented without consuming entanglement. This implies that entanglement is the resource to overcome the restriction to LOCC and that the sole condition a function has to fulfill to be a valid entanglement measure is that it is nonincreasing under LOCC [3,7]. For the bipartite case, a simple criterion for pure state transformations via LOCC has been presented [8]. These results do not only allow one to identify the state $|\Phi^+\rangle \propto \sum_i |ii\rangle$ as the maximally entangled state, which can be transformed into any other bipartite state

deterministically via LOCC, but also allow one to introduce new entanglement measures. Because of the existence of different SLOCC (stochastic LOCC) classes in the multipartite setting [9,10], i.e., the existence of pairs of states that cannot even probabilistically be transformed locally into each other, there does not exist a single state which is *the* optimal resource to overcome LOCC. This is why a set of states, the maximally entangled set (MES) of n subsystems, MES_n, has to be considered [11]. It is the minimal set of states from which any fully entangled n-partite state can be obtained via LOCC.

In order to quantify entanglement, possible LOCC transformations among multipartite states have to be further investigated with the intention to identify new operational entanglement measures. This is precisely the aim of this Letter. We introduce operational entanglement measures for multipartite states (pure or mixed) of arbitrary dimensions [12]. As we are going to show, the measures can be easily computed whenever all possible LOCC transformations are known, as in the case of pure states describing few-partite systems [8,13,14]. The operational character of the new measures allows us to prove very easily that they are indeed nonincreasing under LOCC and admits a generalization to two classes of entanglement measures.

We outline how to compute the new measures for bipartite pure states of arbitrary dimensions. For pure three-qubit systems we derive explicit formulas for them and show that they, together with some well-known bipartite measures, allow us to completely characterize the entanglement contained in the state in an operational way. This characterization shows that the W and GHZ state are the most useful tripartite states regarding state manipulation.

Throughout this Letter, $\sigma_x, \sigma_y, \sigma_z$ denote the Pauli operators and $\mathbb{1}$ the identity operator. When studying possible LOCC transformations we always consider representatives of local unitary (LU)-equivalence classes, as LUs do not alter the entanglement contained in a state and

can obviously always be applied to a state. We say that a state $|\Psi\rangle$ can *reach* a state $|\Phi\rangle$ if there exists a LOCC protocol that transforms $|\Psi\rangle$ into $|\Phi\rangle$ (deterministically). In this case, $|\Phi\rangle$ is *accessible* from $|\Psi\rangle$.

Let us now introduce the new operational entanglement measures. For a given state $|\Psi\rangle$, we denote by $M_a(|\Psi\rangle)$ the set of states that can be accessed via LOCC from $|\Psi\rangle$ and by $M_s(|\Psi\rangle)$ the set of states that can reach $|\Psi\rangle$. The following two magnitudes then occur naturally in the context of possible LOCC transformations: the source volume, $V_s(|\Psi\rangle) = \mu[M_s(|\Psi\rangle)]$, which measures the amount of states that can be used to reach the state $|\Psi\rangle$, and the accessible volume, $V_a(|\Psi\rangle) = \mu[M_a(|\Psi\rangle)]$, which measures the amount of states that can be accessed by $|\Psi\rangle$ via LOCC. Here, μ denotes an arbitrary measure in the set of LU equivalence classes. The underlying idea is that if a state $|\Psi\rangle$ can be reached by many states, i.e., if $M_s(|\Psi\rangle)$ is very large, then the state is not very powerful, as any state in $M_s(|\Psi\rangle)$ could be used for the same purpose and for possibly more applications. On the other hand, if the accessible set is very large, the state is very valuable, as it can be used for any potential application of any state in $M_a(|\Psi\rangle)$.

Because of the operational meaning of M_a and M_s, it is easy to construct now operational entanglement measures. In order to do so, we first show that M_a (M_s) can only become smaller (larger) under LOCC, respectively. Consider a state $|\Psi\rangle$ and any state $|\Phi\rangle_\Psi$ which is accessible from $|\Psi\rangle$ via LOCC. As any state in $M_s(|\Psi\rangle)$ can first be transformed via LOCC into $|\Psi\rangle$ and then to $|\Phi\rangle_\Psi$, it is obvious that $M_s(|\Phi\rangle_\Psi)$ contains $M_s(|\Psi\rangle)$. That $M_a(|\Phi\rangle_\Psi) \subseteq M_a(|\Psi\rangle)$ can be easily verified noting that any state that can be reached from a state $|\Phi\rangle_\Psi$ can, in particular, be reached from a state that can reach $|\Phi\rangle_\Psi$. Hence, any properly normalized and rescaled measure of these sets is indeed an entanglement measure; i.e., it does not increase under LOCC. A possible choice would be $E_a(|\Psi\rangle) = V_a(|\Psi\rangle)/V_a^{\text{sup}}$ and $E_s(|\Psi\rangle) = 1 - V_s(|\Psi\rangle)/V_s^{\text{sup}}$, where V_a^{sup} (V_s^{sup}) denote the maximally accessible (source) volume according to the measure μ. Note that these operational entanglement measures are applicable to arbitrary multipartite systems of any dimension. Moreover, these are valid entanglement measures for mixed states. Note further that $M_s(|\Psi\rangle) = \varnothing$ [implying that $V_s(|\Psi\rangle) = 0$] if and only if the state $|\Psi\rangle$ is in the MES, as these are the only states that cannot be reached by any other state [11]. We elaborate on how these measures can be computed in the case of few-partite pure states below.

The notion of these entanglement measures can be generalized in the following way. Considering an n-partite state, $|\Psi\rangle \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_n}$, one can also measure its entanglement by (i) the amount of $(n-k)$-partite entangled states one can reach from $|\Psi\rangle$, for $k \geq 1$, or (ii) by the amount of reachable states in $\mathbb{C}^{d'_1} \otimes \cdots \otimes C^{d'_n}$, where at least one of the local dimensions d'_i is reduced. Similarly, one can generalize the notion of the source volume to a

whole class of entanglement measures by relating not only elements of the same Hilbert space.

Now, we are going to use these quantities and the previously obtained results on possible LOCC transformations [8,13,15] in order to quantify the entanglement contained in few-body pure states. Let us start by considering the bipartite case. We consider without loss of generality two d-level systems. It is well known that a state $|\Psi\rangle$ can be transformed into a state $|\Phi\rangle$ via LOCC if and only if λ_Ψ is majorized by λ_Φ, i.e., $\lambda_\Psi \preceq \lambda_\Phi$, where λ_Ψ denotes the vector containing the eigenvalues of the single party reduced state of $|\Psi\rangle$ [8]. As the state is normalized, any vector λ_Ψ belongs to a d-dimensional simplex. It has been shown that the set $S(y) = \{x \in \mathbf{R}^d | x \preceq y\}$ is the convex hull of $d!$ points obtained by permuting the components of y [16]. Hence, in this parameter space and using the Lebesgue measure, the source and accessible volume of a state $|\Psi\rangle$ are given by the volume of $S(\lambda_\Psi)$ and $A(\lambda_\Psi) = \{x \in \mathbf{R}^d | \lambda_\Psi \preceq x\}$, respectively (up to a constant factor, see Ref. [17]). In Ref. [17] we present closed formulas for E_s and its generalization. Moreover, we present an algorithm to determine E_a for arbitrary dimension and also explicit formulas for low dimensions, for which the new measures can be used to completely characterize the Schmidt coefficients.

Let us now present a complete characterization of entanglement of an arbitrary pure three-qubit state. In order to understand how the measures E_a and E_s are defined in this case, we give a few remarks. First, we consider only the source and accessible volumes of genuinely entangled three-qubit states (i.e., we do not take into account biseparable states). Second, when we consider LOCC incomparable families of states, such as the W and GHZ SLOCC classes, there is a freedom in choosing different measures μ_1 and μ_2 to compute the volumes for the different families without compromising the behavior under LOCC of the entanglement measures. We exploit this freedom out of mathematical convenience. Last, even when considering LOCC comparable states, there exist states for which the corresponding source or accessible states are in manifolds of different dimensionality. Hence, if we use the same measure μ in both cases, this would assign a zero value to the accessible or source volumes of certain states even though they can indeed reach or be reached by other states, leading to a too coarse grained classification. Even though this would be a legitimate choice, we choose to use here a finer classification by choosing different measures to compute the volumes whenever the corresponding manifolds have different dimensions. Note that this choice, in contrast to the aforementioned one, allows us to compare the relative strength of states whose volumes have the same dimensionality. It should be clear, however, that a state with, e.g., a nonvanishing four-dimensional accessible volume is infinitely more powerful than a state with a three-dimensional accessible volume.

Up to LUs, any state in the W class can be written as [11,15]

$$|\Psi(\vec{x})\rangle = \sqrt{x_0}|000\rangle + \sqrt{x_1}|100\rangle + \sqrt{x_2}|010\rangle + \sqrt{x_3}|001\rangle,$$
(1)

where $x_1, x_2, x_3 > 0$, $x_0 \geq 0$, and $\sum_{i=0}^{3} x_i = 1$. Note that any state $|\Psi(\vec{x})\rangle$ can be represented by the corresponding vector $\vec{x} = (x_1, x_2, x_3)$ within a three-dimensional simplex $\mathcal{S}_3$ (see Fig. 1) [15]. As shown in Ref. [15], $|\Psi(\vec{x})\rangle$ can be transformed into $|\Psi(\vec{y})\rangle$ via LOCC if and only if $x_i \geq y_i$, $\forall i \in \{1,2,3\}$. Because of that, it can be easily verified that the MES in the W class, the W MES, is the set of states with $x_0 = 0$. These states cannot be obtained from any other state, but any state in the W class can be obtained from some state with $x_0 = 0$ [11].

Within the parameter space explained above, the accessible volume and the source volume of an arbitrary state $|\Psi(\vec{x})\rangle$ can be easily shown to be (see also Fig. 1)

$$V_a(|\Psi(\vec{x})\rangle) = x_1 x_2 x_3, \qquad V_s(|\Psi(\vec{x})\rangle) = \frac{x_0^3}{6}. \quad (2)$$

As mentioned above, it follows already from their definition that the corresponding measures, $E_a(|\Psi(\vec{x})\rangle) = 27 V_a(|\Psi(\vec{x})\rangle)$ and $E_s(|\Psi(\vec{x})\rangle) = 1 - 6 V_s(|\Psi(\vec{x})\rangle)$, are entanglement measures. A fact that can be particularly easily verified for the W class using Eq. (2) and the fact that no x_i can be increased via LOCC. Note that the W state maximizes both new measures, with $E_a(|W\rangle) = 1$ and $E_s(|W\rangle) = 1$. Hence, the W state is the state that reaches the most other states deterministically via LOCC. It can therefore be regarded as the most useful state in the W class.

Let us now characterize the entanglement contained in a state in the W class. Because of the simplicity of this class,
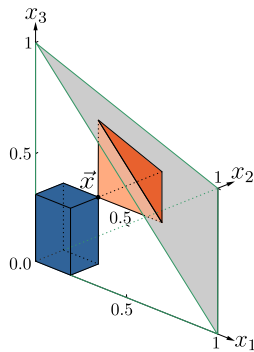

FIG. 1 (color online). Any state $|\Psi(\vec{x})\rangle$ is uniquely represented by $\vec{x}$ in the interior of the simplex $\mathcal{S}_3$. The source (tetrahedron) and the accessible (cuboid) volume of $|\Psi(\vec{x})\rangle$ are depicted. The light surface corresponds to the states in the MES. Biseparable (fully separable) states, for which exactly one (at least two) x_i is (are) zero, are represented by points on the white surface of $\mathcal{S}_3$, respectively.

only bipartite entanglement measures, e.g., the three bipartite entanglement between party i and the remaining parties, measured with, e.g., the squared concurrence [18], $C_i(|\Psi(\vec{x})\rangle) = 4x_i(1 - x_i - x_0)$, are required to uniquely characterize the state (up to LUs). However, one could also employ the new measures and any of the bipartite measures for this purpose. In fact, as any three measures of the set $\{C_1(|\Psi\rangle), C_2(|\Psi\rangle), C_3(|\Psi\rangle), E_a(|\Psi\rangle), E_s(|\Psi\rangle)\}$ are independent, we have that a state in the W class is uniquely determined by any three of these operational entanglement measures.

Note that for any state in the W MES it holds that $V_s(|\Psi(\vec{x})\rangle) = 0$. Moreover, for these states we have $C_i(|\Psi(\vec{x})\rangle) = 4x_i(1 - x_i)$. Hence, they can be easily characterized by any two of these bipartite measures or by any bipartite measure and $E_a(|\Psi(\vec{x})\rangle)$. Note that the characterization via operational entanglement measures of arbitrary states in the W class presented here can be easily generalized to n-qubit systems [17].

Let us now investigate the more complicated GHZ class. A state in the GHZ class can be written (up to LUs) as [11]

$$|\Psi(\boldsymbol{g}, z)\rangle = g_x^1 \otimes g_x^2 \otimes g_x^3 P_z |\text{GHZ}\rangle, \quad (3)$$

with $(g_x^i)^\dagger g_x^i = \frac{1}{2}\mathbb{1} + g_i \sigma_x$, $g_i \in [0, 1/2)$ $\forall i, \boldsymbol{g} = (g_1, g_2, g_3)$, $P_z = \text{diag}(z, 1/z), z \in \mathbb{C}$, with $|z| \leq 1$. As shown in Ref. [11], a state is in GHZ MES if and only if $z = 1$ or $z = i$ and either none of the g_i's vanish or all of them vanish, corresponding to the GHZ state.

In Ref. [13] the necessary and sufficient conditions for the existence of a LOCC transformation from a state $|\Psi(\boldsymbol{g}, z)\rangle$ to another state $|\Psi(\boldsymbol{h}, z')\rangle$ were obtained. As shown there, the absolute value of z can be changed by LOCC independently of the other parameters only if at least one of the parameters g_i vanishes, in which case $|z|$ can be arbitrarily decreased. As in this case, different LOCC transformations are possible, and we treat the cases (A) $g_i \neq 0$ for all i and (B) at least one of the parameters g_i vanishes separately.

Let us first consider case (A). Expressing the conditions for the existence of a LOCC transformation [13] from $|\Psi(\boldsymbol{g}, z)\rangle$ to $|\Psi(\boldsymbol{h}, z')\rangle$ [see Eq. (3)], we obtain (i) $g_i \leq h_i$ $\forall i$, (ii) $g_1 g_2 g_3 / (h_1 h_2 h_3) = [\text{Re}(z'^2)/(|z'|^4 + 1)][(|z|^4 + 1)/\text{Re}(z^2)] = [\text{Im}(z'^2)/(|z'|^4 - 1)][(|z|^4 - 1)/\text{Im}(z^2)]$.

Note that condition (ii) constitutes generically two independent equalities. However, in case the numerator and/or the denominator of one ratio vanishes, different conditions have to hold (see Ref. [19]).

We now present a characterization of the entanglement contained in an arbitrary state in the GHZ class.

For this purpose, we first consider states that are neither in GHZ MES nor any g_i vanishes, i.e., case (A) with $z \neq 1, i$. The other cases are treated below. The accessible and the source volume are given by (see Ref. [19])
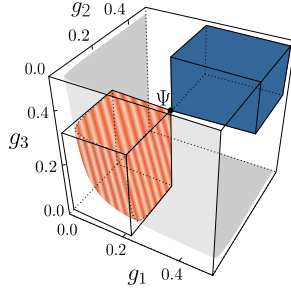
FIG. 2 (color online). The source (shaded volume) and accessible (cuboid) volume of the state $|\Psi\rangle$ with parameters $g_1 = 0.22$, $g_2 = 0.26$, $g_3 = 0.32$, and $z = 0.1e^{i2.68}$ are depicted. The states in the MES, which fulfill condition (ii), are on the light area.

$$V_a(|\Psi(\mathbf{g}, z)\rangle) = (1/2 - g_1)(1/2 - g_2)(1/2 - g_3), \quad (4)$$

$$V_s(|\Psi(\mathbf{g}, z)\rangle)$$
$$= G\left(1 + f_z\left\{\log(f_z)\left[1 - \frac{1}{2}\log(f_z)\right] - 1\right\}\right), \quad (5)$$

with $f_z = 2|\mathrm{Re}(z^2)|/(1 + |z|^4)$ and $G = g_1 g_2 g_3$ (see Fig. 2).

We show in Ref. [19] that $E_a(|\Psi(\mathbf{g}, z)\rangle) = 8V_a(|\Psi(\mathbf{g}, z)\rangle)$ and $E_s(|\Psi(\mathbf{g}, z)\rangle) = 1 - 8V_s(|\Psi(\mathbf{g}, z)\rangle)$ together with the three bipartite entanglement measures and one additional bit, that provides information about a specific state in the source set, uniquely determine the five parameters $z = re^{i\phi}, g_1, g_2, g_3$ and therefore uniquely characterize the entanglement of the states up to complex conjugation (taken with respect to the computational basis) and LUs [21] (see Ref. [19] for details).

In Ref. [19] we show that the states where at least one g_i vanishes can be treated similarly, and that there the entanglement of the states is uniquely determined by the five operational entanglement measures.

It remains to consider the states in GHZ MES, which constitute a three parameter family. In this case only condition (i) and the first equation in condition (ii) have to be fulfilled, which implies that only one parameter of the accessible states is fixed via condition (ii). Hence, for $|\Psi_{\mathrm{MES}}\rangle$, a state in MES$_3$, we obtain the four-dimensional accessible volume

$$V_a(|\Psi_{\mathrm{MES}}\rangle)$$
$$= \int_{g_1}^{1/2} \int_{g_2}^{1/2} \int_{|g_3|}^{1/2} \int_{\sqrt{(H/G) - \sqrt{(H/G)^2 - 1}}}^{1} dr\, dh_3\, dh_2\, dh_1,$$
$$(6)$$

with $G = g_1 g_2 |g_3|$, $H = h_1 h_2 h_3$. Putting the lower limits in Eq. (6) to zero, we find $V_a^{\sup} = 1/8$. As the GHZ state fulfills $g_i = 0 \quad \forall\, i$, we have $E_a(|\mathrm{GHZ}\rangle) = 1$, and therefore the GHZ state is the state that reaches the most other

states deterministically. The entanglement of a state in GHZ MES, for which $V_s(|\Psi_{\mathrm{MES}}\rangle) = 0$, can be similarly easily characterized, as it was possible in the W class (see Ref. [19]).

In summary, we have introduced two novel classes of operational entanglement measures, which are applicable to arbitrary multipartite pure or mixed states. We then demonstrated how these measures can be computed for the simplest pure multipartite case (three qubits) and showed that they can be used to completely characterize the entanglement contained in a three-qubit state. In Ref. [17] the new measures and its generalizations are determined for the bipartite setting of low dimension and the four-qubit case. It would be interesting to develop further the analysis of LOCC convertibility among mixed states in order to compute our measures in this case. In addition to that and further extensions of this approach (e.g., approximate LOCC transformations, multicopy case), it would also be appealing to connect our measures with different quantum information protocols and condensed matter phenomena, which we leave for future research.

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, New York, 2000).
[2] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008), and references therein.
[3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009), and references therein.
[4] F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004).
[5] M. Christandl and A. Winter, J. Math. Phys. (N.Y.) **45**, 829 (2004).
[6] E. Chitambar, D. Leung, L. Mančinska, M. Ozols, and A. Winter, Commun. Math. Phys. **328**, 303 (2014).
[7] Note that sometimes entanglement quantifiers that are nonincreasing on average under LOCC have been considered (see Ref. [3]).
[8] M. A. Nielsen, Phys. Rev. Lett. **83**, 436 (1999).
[9] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).
[10] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, Phys. Rev. A **65**, 052112 (2002).
[11] J. I. de Vicente, C. Spee, and B. Kraus, Phys. Rev. Lett. **111**, 110502 (2013).
[12] Note that the fact that the measures are applicable for any dimension is crucial, as LOCC might increase the dimension of the considered Hilbert space.
[13] S. Turgut, Y. Gül, and N. K. Pak, Phys. Rev. A **81**, 012317 (2010).

[14] Note that the LOCC convertibility among mixed states is only partially known even in the case of bipartite systems [22].

[15] S. Kintaş and S. Turgut, J. Math. Phys. (N.Y.) **51**, 092202 (2010).

[16] R. Rado, J. Lond. Math. Soc. **s1-27**, 1 (1952).

[17] D. Sauerwein, K. Schwaiger, M. Cuquet, J. I. de Vicente, and B. Kraus, arXiv:1508.02524.

[18] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[19] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.115.150502, which includes Ref. [20], for the detailed calculation of all volumes and for full proofs of the characterization of entangled states with the set of measures.

[20] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, Phys. Rev. A **63**, 012307 (2000).

[21] Note that complex conjugation of a state corresponds to a redefinition of the complex unit and therefore cannot alter any operational entanglement measure [23], if one considers only the state of interest $|\Psi\rangle$.

[22] P. Hayden, B. M. Terhal, and A. Uhlmann, arXiv:quant-ph/0011095.

[23] B. Kraus, Phys. Rev. Lett. **104**, 020504 (2010); Phys. Rev. A **82**, 032121 (2010).